Temporal Convergence and Stability Assessment of the Generalized Finite Element Method for Muli-Scale Field Problems

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Objectives

Goal: Expand research on multi-scale numerical modeling strategies to transient heat transfer problems with highly localized loading conditions through establishing implementation strategies and characterizing the temporal convergence behavior of GFEM with general time-integrators

1. Assess the temporal convergence, accuracy and stability of the GFEM for a variety of spatial domains and parameters

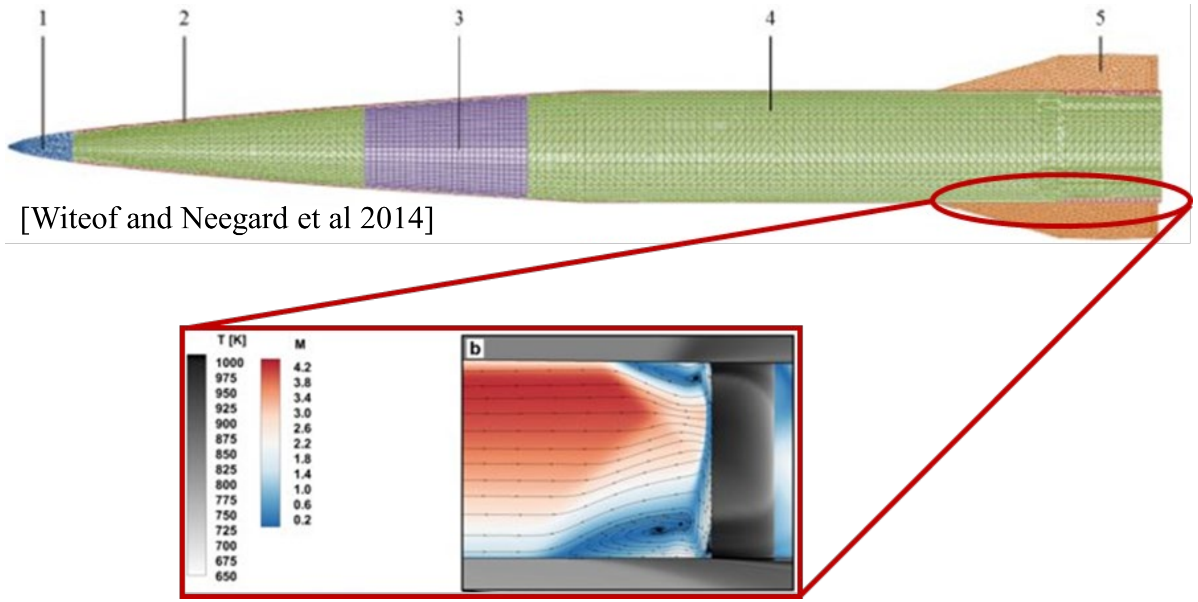
2. Investigate computational savings compared to standard methods

3. Identify nuances and implementation strategies of the GFEM

Challenges with Hypersonic Analysis

- Modern day engineering problems in hypersonic vehicle design are dominated by heat transfer
  - Fine-scale and transient loading conditions
  - Coupled, multi-physic interactions
- Current multi-scale modeling strategies lack power to resolve all spatio-temporal scales on a global level
  - Fine meshes needed for spatial gradients
  - Broad regions of refinement for transient features
  - Small time steps to maintain temporal stability
- High-fidelity solutions often require large amounts of CPU power, time and memory

Question: How can we simultaneously capture fine-scale features and global phenomena within a multi-physics simulation efficiently?



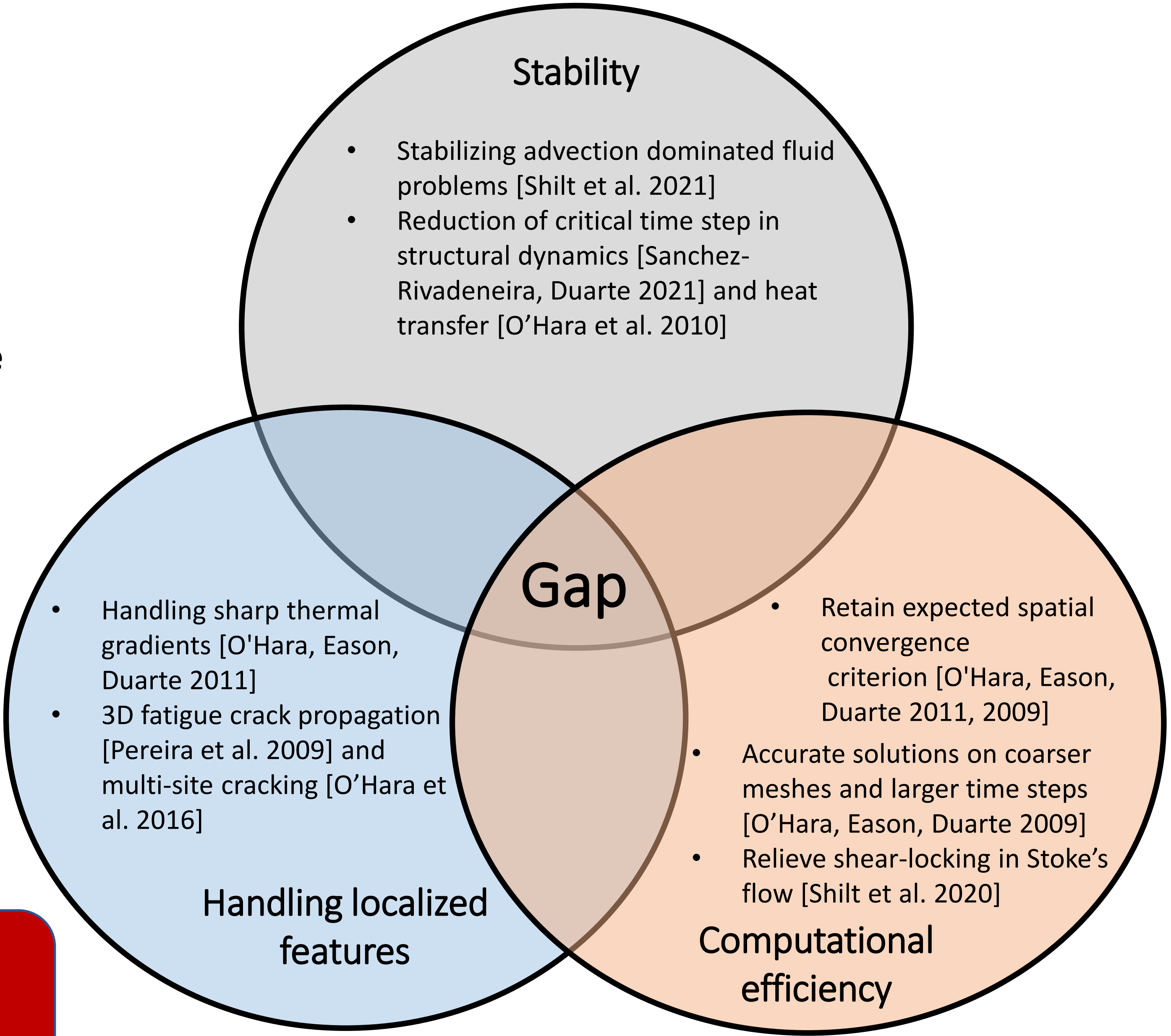
[Witeof and Neegard et al 2014]

Image courtesy of Jon Willem, The Ohio State University

- Above is the flow and temperature field over a torque tube experiencing sharp thermal gradients on a small scale due to shock-boundary layer interactions
- Accurately resolving local features across all spatio-temporal scales, while avoiding local mesh refinement and advanced multi-scale methods, is **essential for practical modeling** of multi-physic simulations

Motivation

- Heat transfer in extreme environments is multi-scale and coupled with fluids and structural analysis
- Mathematically, heat transfer is a scalar equation
  Implementation differs from vectoral analysis of fluids and tensorial analysis of structures
- Enabling solutions of heat transfer problems in extreme conditions is **essential for high-speed vehicle design**
- The GFEM incorporates solution-tailored shape functions to alleviate the need for local mesh refinement
- Current work has focused on ability for GFEM to capture localized features efficiently, spatial convergence criteria, and stability with focus on fluids and structural problems in multi-scale environments
- Lack of research to extend these concepts to heat transfer and transient analyses of general time integrators has led to a gap in knowledge



Hypothesis: GFEM can enable high-fidelity solutions of extreme multi-scale heat transfer problems that are currently prohibitive in the context of multi-physics simulations

Overview of the GFEM

Modifies FEM framework to introduce solution tailored "enrichments" into standard FEM space

FEM Approximation Space

$\{\varphi_\alpha(x)\}_{\alpha=1}^{N}$

*

Enrichment Space

$\{L_{\alpha j}(x)\}_{\mathrm{j}=1}^{\mathrm{m}_\alpha}$

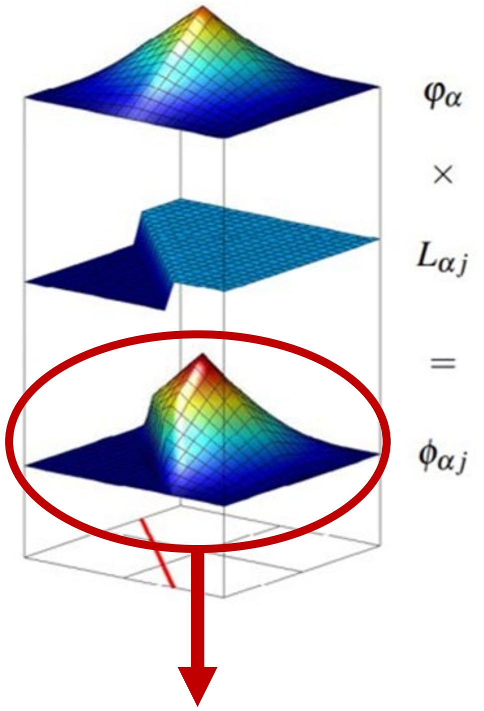
- Can be any function; typically derived from *a priori* knowledge

→

GFEM Enrichment Space

$\phi_{\alpha j} = \{\varphi_\alpha(x) L_{\alpha j}(x)\}_{\mathrm{j}=1}^{\mathrm{m}_\alpha}$

- GFEM shape functions formed through a product of standard FEM shape functions and enriched trial space



Includes discontinuity into shape function

Final approximation: $u^h(x) = \sum_{\alpha \in I_h} \varphi_\alpha(x) \sum_{j=1}^{m_\alpha} \tilde{u}_{\alpha j} L_{\alpha j}(x)$

GFEM elegantly handles fine-scale features by directly introducing these features into the computational domain, alleviating local mesh refinement

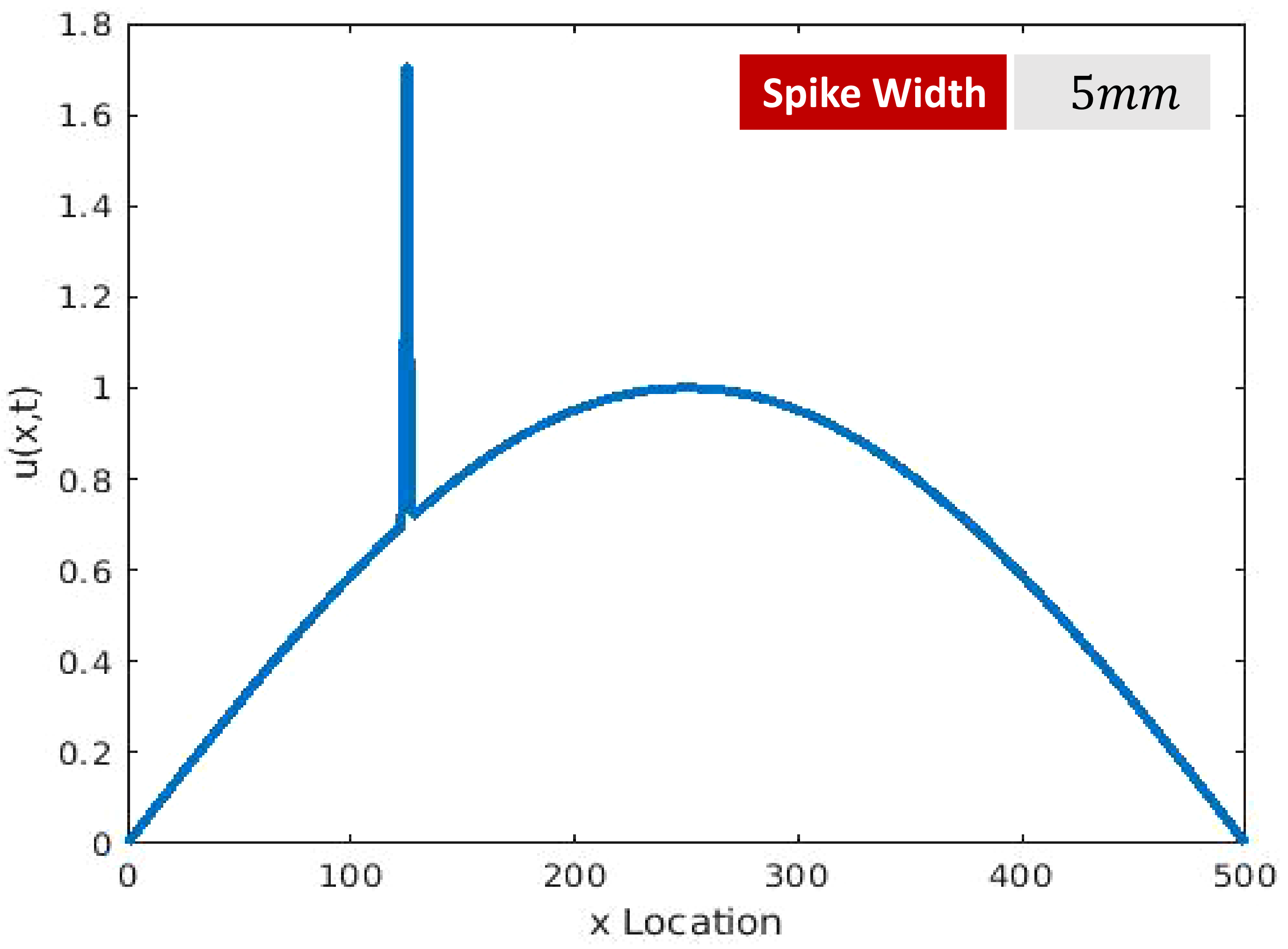
Model Problem from Merle and Dolbow 2002

$$Q(x,t) = \rho c \frac{\partial u}{\partial t}(x,t) - k \frac{\partial^2 u}{\partial x^2}(x,t)$$

$$u(x,t) = \left(e^{-\gamma (x - x_{front}(t))^2} + \sin\left(\frac{\pi x}{L}\right) \right) e^{-t}$$

$$x_{front}(t) = x_0 + Vt$$

Property Definitions	
ρc	$\left(\frac{\pi}{L}\right)^2 \frac{J}{K * mm^3}$
k	$1 \frac{W}{K * mm}$
L	$500mm$
x_0	$125mm$
γ	1
V	$250 \frac{mm}{s}$

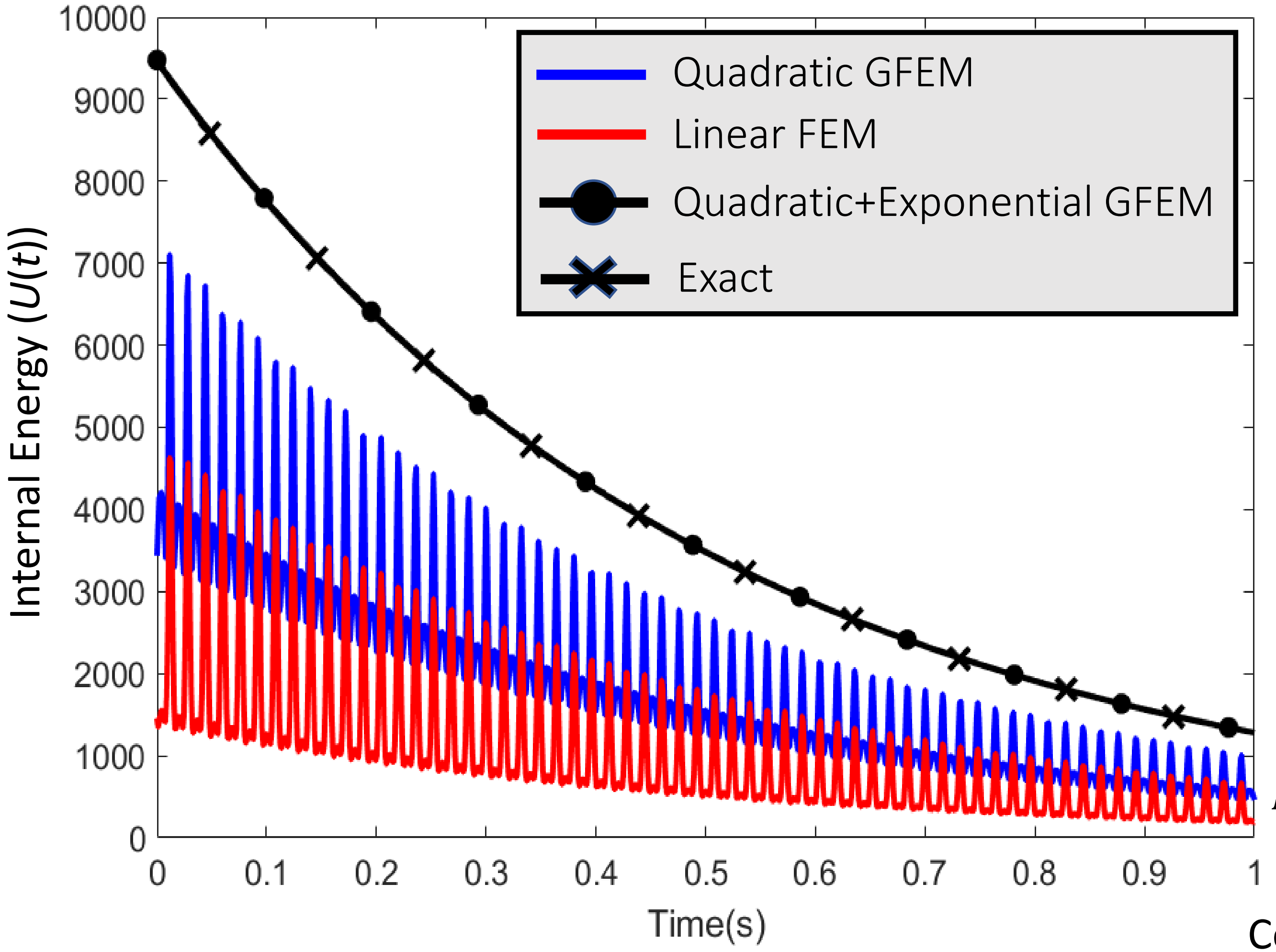


[Merle and Dolbow 2002]

$$\Omega = \{0 < x < 500mm, 0 < y < 250mm, 0 < z < 30mm\}$$

Internal Energy Profile for Different Approximation Spaces

Number of Elements	Element Dimensions (4-point Tetrahedron)	Time Step
1500	$4mm \times 125mm \times 30mm$ $(Width \times Depth \times Height)$	0.0009766s (1024 steps)



- Energy profile prediction depends on ability for approximation space to capture the spike in loading/temperature
  - Low order FEM and GFEM cannot capture this fine-scale feature (5mm) **on a coarse mesh of 4mm wide elements**
  - Exponential GFEM does capture the gradient **without mesh refinement**
- **GFEM enables accurate multi-scale solutions on coarse meshes**

$$U(t) = \int_{\Omega} k\big(\nabla u(x,t) \cdot \nabla u(x,t)\big)\, d\Omega$$

Quadratic+Exponential GFEM:

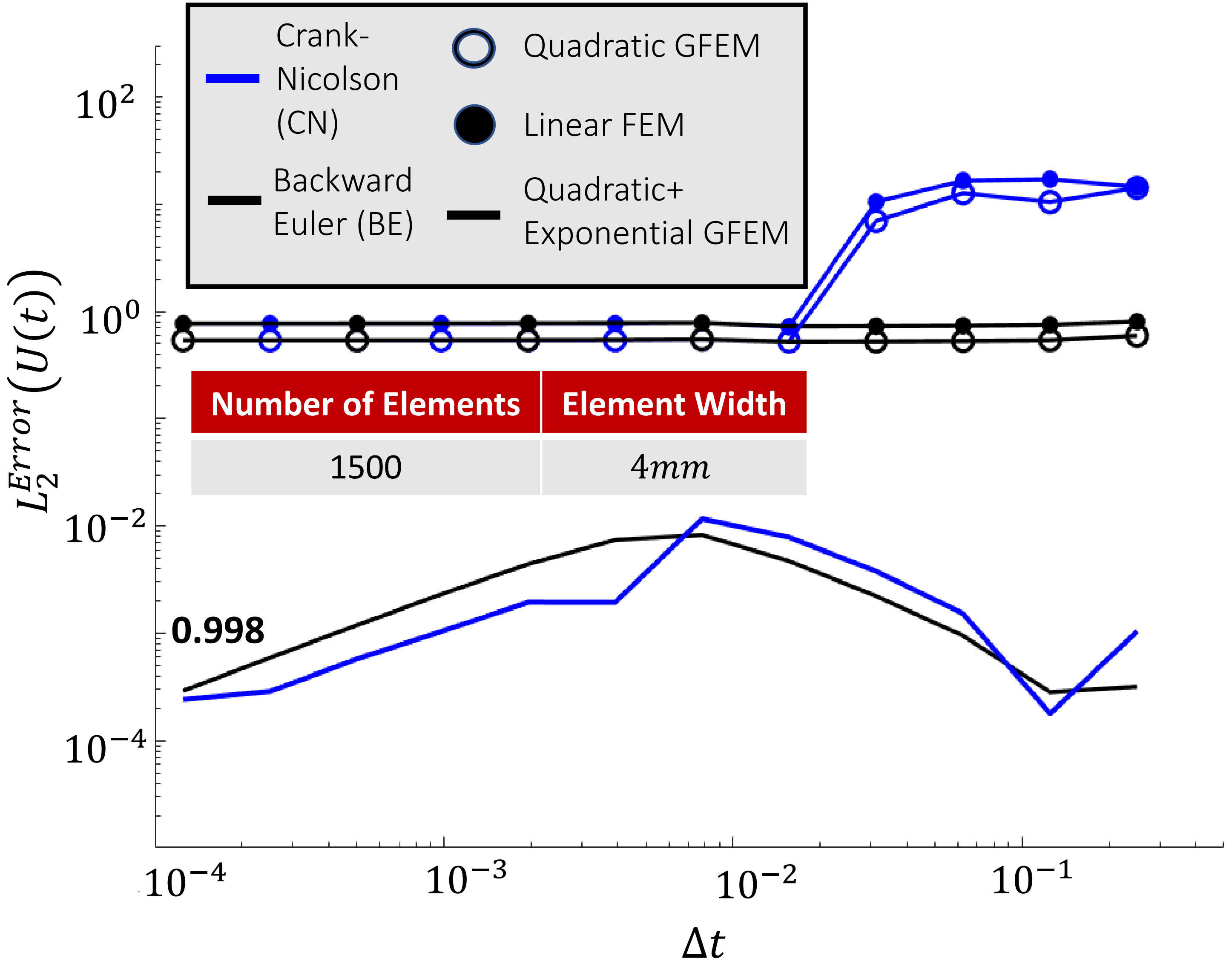
$$L_{\alpha j} = \left\{ 1, \frac{x - x_\alpha}{h}, e^{-(x - x_{front}(t))^2} \right\}$$

Corresponds to FEM DOF

Quadratic GFEM:

$$L_{\alpha j} = \left\{ 1, \frac{x - x_\alpha}{h} \right\}$$

Temporal Convergence with Different Time-Integrators



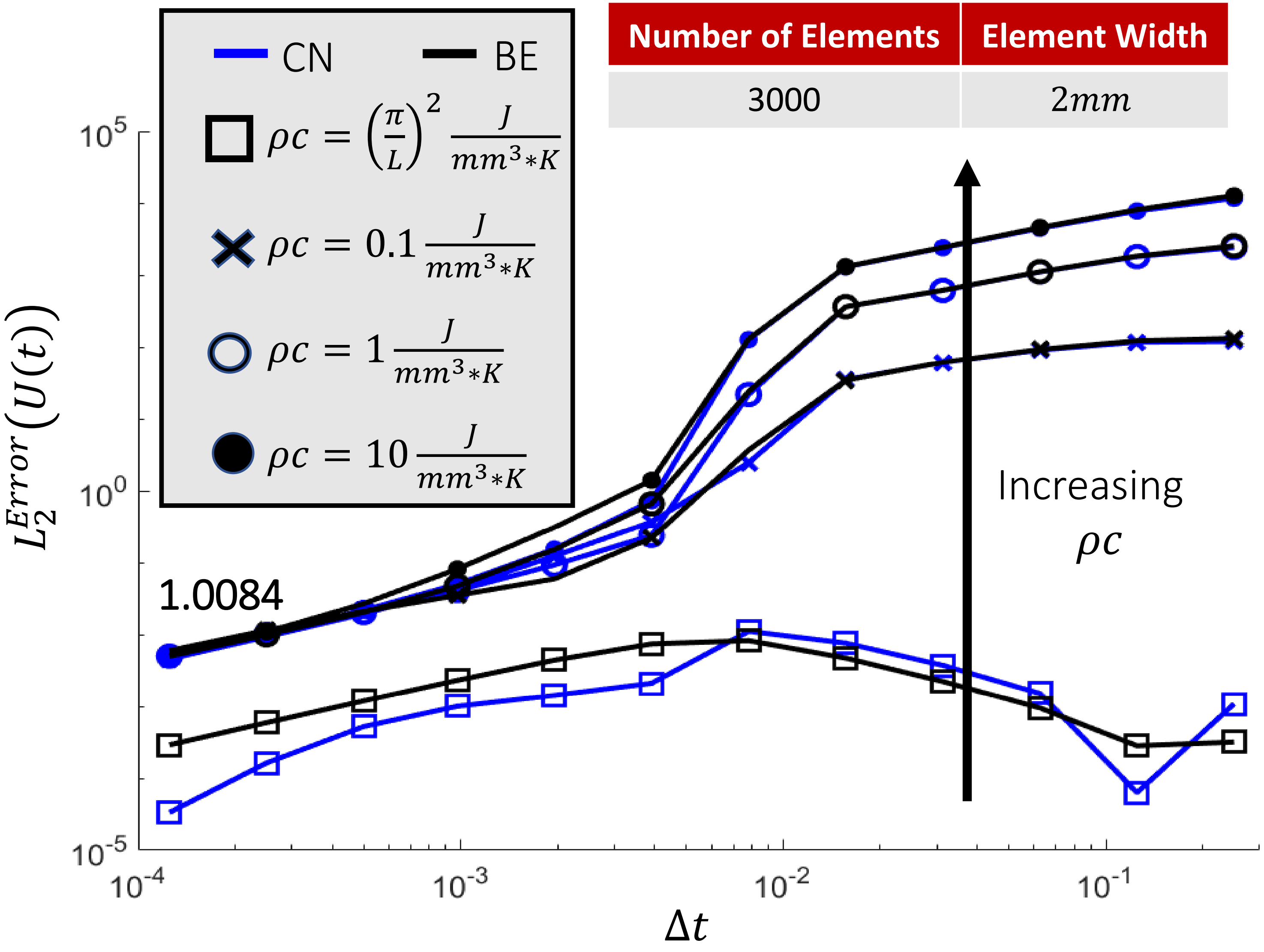
Number of Elements	Element Width
1500	$4mm$

- Shown left are accurate multi-scale solutions with **larger time steps**
- GFEM using **solution-tailored, exponential enrichments** obtains temporal convergence
- **Lower order approximations** obtain no temporal convergence
  - Spatial error dominates due to not capturing the spike
- **GFEM enables temporally convergent multi-scale solutions**

$$L_2^{\text{Error}}(\text{U(t)}) = \left\{ \frac{\sum_n (U_{exact}(t^n) - U_h\,(t^n))^2}{\sum_n (U_{exact}(t^n))^{\,2}} \right\}^{\frac{1}{2}}$$

Effect of Temporal Gradient Strength on Convergence

Number of Elements	Element Width
3000	$2mm$



- The volumetric heat capacity (ρc) **controls the magnitude** of the temporal gradient
  - Analytical solution is **independent** of this
- Stronger gradient corresponds to a **substantial increase** in error at large time steps
- Both time integrators approach the **same convergence rates and error levels** when the time-scale of the moving singularity is resolved at larger ρc
- **Stronger temporal gradients increases error for both methods, but degrades convergence of the Crank-Nicolson method**

Conclusions and Future Work

- Temporal convergence obtained with time-dependent, solution tailored enrichments on coarse grids on a 3D domain
- Convergence study demonstrates GFEM can achieve accurate solutions on coarse meshes and larger time steps in multi-scale problems
- Strengthening the temporal gradient increases error and induces temporal oscillations (not shown) in both methods while degrading the performance of Crank-Nicolson
- Initial stability study (not shown) and convergence results indicate potential for GFEM to increase critical time steps with solution-tailored enrichments
- Results provide improved confidence in the ability of GFEM to enable simulations of design critical multi-scale, multi-physics problems of hypersonic systems

- Future Work
  - In-depth error analysis to determine the temporal behavior of Crank-Nicolson and Backwards Euler in the GFEM framework
  - Explore the use of global-local enrichments for the model problem
  - Analyze the stability and convergence behavior of the Forward Euler method in 3D
  - Investigate the convergence behavior of the temporal-integrators when the singularity is not aligned with the primary mesh direction